

PARAHERMITIAN AND PARAQUATERNIONIC MANIFOLDS

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ABSTRACT. A set of canonical parahermitian connections on an almost paraHermitian manifold is defined. ParaHermitian version of the Apostolov-Gauduchon generalization of the Goldberg-Sachs theorem in General Relativity is given. It is proved that the Nijenhuis tensor of a Nearly paraKähler manifolds is parallel with respect to the canonical connection. Salamon's twistor construction on quaternionic manifold is adapted to the paraquaternionic case. A hyper-paracomplex structure is constructed on Kodaira-Thurston (properly elliptic) surfaces as well as on the Inoe surfaces modeled on Sol_1^4 . A locally conformally flat hyper-paraKähler (hypersymplectic) structure with parallel Lee form on Kodaira-Thurston surfaces is obtained. Anti-self-dual non-Weyl flat neutral metric on Inoe surfaces modeled on Sol_1^4 is presented. An example of anti-self-dual neutral metric which is not locally conformally hyper-paraKähler is constructed.

Key words: indefinite neutral metric, product structure, self-dual neutral metric, paraHermitian, paraquaternionic, Nearly paraKähler manifold, hyper-paracomplex, hyper-paraKähler (hypersymplectic) structures, twistor space.

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1. INTRODUCTION

We study the geometry of structures on a differentiable manifold related to the algebra of paracomplex numbers as well as to the algebra of paraquaternions together with a naturally associated metric which is necessarily of neutral signature. These structures lead to the notion of almost paraHermitian manifold, in even dimension, as well as to the notion of almost paraquaternionic and hyper-paraHermitian manifolds in dimensions divisible by four. Some of these spaces, hyper-paracomplex and hyper-paraHermitian manifolds, become attractive in theoretical physics since they play a role in string theory [55, 40, 44, 11, 41] and integrable systems [26].

Almost paraHermitian geometry is a topic with many analogies with the almost Hermitian geometry and also with differences. In the present note we show that a lot of local and some of the global results in almost Hermitian manifolds carry over, in the appropriately defined form, to the case of almost paraHermitian spaces.

We define a set of canonical paraHermitian connections on an almost paraHermitian manifold and use them to describe properties of 4-dimensional paraHermitian and 6-dimensional Nearly paraKähler spaces.

We present a paraHermitian analogue of the Apostolov-Gauduchon generalization [6] of the Goldberg-Sachs theorem in General Relativity (see e.g. [57]) which relates the Einstein condition to the structure of the positive Weyl tensor in dimension 4. Namely, we prove

Theorem 1.1. *Let (M, g, P) be a 4-dimensional paraHermitian manifold. Let W^+ be the self-dual part of the Weyl tensor and θ be the Lee 1-form. The following conditions are equivalent:*

- a) *The 2-form $d\theta$ is anti-self-dual, $d\theta^+ = 0$;*
- b) *$W_2^+ = 0$, equivalently, the fundamental 2-form is an eigen-form of W^+ ;*
- c) *$(\delta W^+)^- = 0$, equivalently, $(\delta W)(X^{1,0}; Y^{1,0}, Z^{1,0}) = 0$.*

Corollary 1.2. *Assume that the Ricci tensor ρ of a paraHermitian 4-manifold is P -anti-invariant, $\rho(PX, PY) = -\rho(X, Y)$. Then $d\theta$ is anti-self-dual 2-form, $d\theta^+ = 0$.*

In particular, on a paraHermitian Einstein 4-manifold the fundamental 2-form is an eigen-form of the positive Weyl tensor.

It turns out that any conformal class of neutral metrics on an oriented 4-manifold is equivalent to the existence of a local almost hyper-paracomplex structure, i.e. a collection of anti-commuting almost complex structure and almost para-complex structure. Using the properties of the Bismut connection, we derive that the integrability of the almost hyper-paracomplex structure leads to the anti-self-duality of the corresponding conformal class of neutral metrics (Theorem 6.2). Applying this result to invariant hyper-paracomplex structure on 4-dimensional Lie groups [4, 22] we find explicit anti-self-dual non Weyl flat neutral metrics on some compact 4-manifolds. Some of these metrics seem to be new.

We apply our considerations to Kodaira-Thurston complex surfaces modeled on $S^1 \times \widetilde{SL(2, \mathbb{R})}$ (properly elliptic surfaces) as well as to the Inoue surfaces modeled on Sol_1^4 in the sense of [65]. These surfaces do not admit any (para) Kähler structure [65, 18, 58]. It is also known that these surfaces do not support a hyper-complex structure [49, 23].

In contrast, we obtain

Theorem 1.3. *The Kodaira-Thurston surfaces $M = S^1 \times (\widetilde{SL(2, \mathbb{R})}/\Gamma)$ admit a hyper-paracomplex structure. The corresponding hyper-paraHermitian structure has ∇^g -parallel Lee form and is locally (not globally) conformally equivalent to a flat hyper-paraKähler (hy-persymplectic) structure.*

Theorem 1.4. *The Inoe surfaces modeled on Sol_1^4 admit a hyper-paracomplex structure. The corresponding neutral metric is anti-self-dual non-Weyl flat. The para-hermitian structure is locally (but not globally) conformally hyper-paraKähler (hypersymplectic).*

The Inoe surfaces modeled on Sol_1^4 are compact solvmanifolds. A compact 4-dimensional solvmanifold S can be written, up to double covering, as G/Γ where G is a simply connected solvable Lie group and Γ is a lattice of G and all compact four-dimensional solvmanifolds admitting a complex structure are classified recently in [38]. Except the Inoe surfaces modeled on Sol_0^4 , all other compact four-dimensional solvmanifolds admitting a complex structure support also an hyper-paracomplex structure due to the results in [58, 46, 29] and Theorem 1.4. It is also shown in [38] that every complex structure on a compact 4-dimensional solvmanifold is the canonical complex structure induced from the left-invariant complex structure on the solvable Lie group G . The four-dimensional Lie algebras admitting hyper-paracomplex structure are classified in [22]. A glance on Lie algebras listed in [22] leads to the conclusion that the Inoe surfaces modeled on Sol_0^4 do not admit a hyper-paracomplex structure induced from a left-invariant hyper-paracomplex structure on the solvable Lie group Sol_0^4 .

In view of Theorem 6.2 and Theorem 1.4, a naturally arising question is whether the existence of a self-dual neutral metric distinguishes the Inoe surfaces modeled on Sol_1^4 and the Inoe surfaces modeled on Sol_0^4 , i.e. whether there exists a hyper-paracomplex structure on the Inoe surfaces modeled on Sol_0^4 .

We construct an anti-self-dual neutral metric which is not locally conformally hyper-paraKähler (hypersymplectic). We adapt the Ashtekar at all [7] formulation of the self-duality Einstein equations to the case of neutral metric and modify the Joyce's construction [45] of hyper-complex structure from holomorphic functions to get hyper-paracomplex structure.

Some properties of hyper-paracomplex and hyper-parahermitian structures in higher dimensions are treated in [42, 43].

We prove that the Nijenhuis tensor of a Nearly paraKähler manifold is parallel with respect to the canonical connection. In dimension six, we show that these spaces are Einsteinian but the Ricci-flat case can not be excluded. This is in contrast with the case of Nearly Kähler 6-manifolds which are Einsteinian with positive scalar curvature. We involve twistor machinery to obtain examples of Nearly paraKähler manifolds. We adapt Salamon's twistor construction on quaternionic manifold [60, 61, 62] to the paraquaternionic situation. We consider the reflector space of a paraquaternionic manifold as a higher dimensional analogue of the reflector space of a 4-dimensional manifold with a metric of neutral signature described in [44]. We show that the reflector space of an Einstein self-dual non-Ricci flat 4 manifold as well as the reflector space of a paraquaternionic Kähler manifold admit both Nearly paraKähler and almost paraKähler structures. We present homogeneous as well as non locally homogeneous examples of 6-dimensional almost paraKähler and Nearly paraKähler manifolds. However, all our examples of Nearly paraKähler 6-manifolds are Einstein spaces with non-zero scalar curvature. To the best of the author's knowledge there are no known examples of Ricci flat 6-dimensional Nearly paraKähler manifolds.

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2. PRELIMINARIES

Let V be a real vector space of even dimension $2n$. An endomorphism $P : V \rightarrow V$ is called a *paracomplex structure* on V if $P^2 = 1$ and the eigenspaces V^{+1}, V^{-1} corresponding to the eigenvalues 1 and -1 , respectively are of the same dimension n , $V = V^{+} \oplus V^{-}$. Consider the algebra

$$\mathbb{A} = \{x + \epsilon y, x, y \in \mathbb{R}, \epsilon^2 = 1\}$$

of paracomplex numbers over \mathbb{R} . As in the ordinary complex case, \mathbb{A}^n is identified with (\mathbb{R}^{2n}, P) , where $Pv = \epsilon v$. P is called *the canonical paracomplex structure* on \mathbb{R}^{2n} .

The notions of (almost) paracomplex, paraHermitian, para-holomorphic, etc., objects are defined in the usual way over the paracomplex numbers \mathbb{A} , instead of the complex numbers \mathbb{C} . A survey on paracomplex geometry is presented in [24].

A $(1,1)$ -tensor field P on an $2n$ -dimensional smooth manifold M is said to be an *almost product structure* if $P^2 = 1$. In this case the pair (M, P) is called *almost product manifold*. An *almost paracomplex manifold* is an almost product manifold (M, P) such that the two eigenbundles $T^{+}M$ and $T^{-}M$ associated with the two eigenvalues ± 1 of P have the same rank. Equivalently, a splitting of the tangent bundle $TM = TM^{+} \oplus TM^{-}$ of the subbundles TM^{\pm} of the same fiber dimension is called an almost paracomplex structure. A smooth section of TM^{+} is called *$(1,0)$ -vector field* while a smooth section of TM^{-} is said to be *$(0,1)$ -vector field* with respect to the almost paracomplex structure. Such a structure may alternatively be defined as a G -structure on M with structure group $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$.

The Nijenhuis tensor N of P is defined by [66]

$$4N(X, Y) = [PX, PY] + [X, Y] - P[PX, Y] - P[X, PY].$$

The structure P is said to be *paracomplex* if $N = 0$ [51] which is equivalent to the distributions on M defined by TM^{\pm} to be both completely integrable [47]. The paracomplex manifold can also be characterized by the existence of an atlas with paraholomorphic coordinate maps i.e. the coordinate maps satisfying the para-Cauchy-Riemann equations [51] (see also [47]).

An *almost paraHermitian manifold* (M, P, g) is a smooth manifold endowed with an almost paracomplex structure P and a pseudo-Riemannian metric g compatible in the sense that

$$g(PX, Y) + g(X, PY) = 0.$$

It follows that the metric g is *neutral*, i.e. it has signature (n, n) and the eigenbundles TM^{\pm} are totally isotropic with respect to g . Equivalently, an almost paraHermitian manifold is a smooth manifold whose structure group can be reduced to the real representation of the para-unitary group

$$U(n, \mathbb{A}) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix}, A \in GL(n, \mathbb{R}) \right\}$$

isomorphic to $GL(n, \mathbb{R})$.

Let $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n} = Pe_1, \dots, Pe_n$ be an orthonormal basis and denote $\epsilon_i = \text{sign}(g(e_i, e_i)) = \pm 1$, $\epsilon_i = 1, i = 1, \dots, n$, $\epsilon_j = -1, j = n+1, \dots, 2n$.

The fundamental 2-form F of an almost paraHermitian manifold is defined by

$$F(X, Y) = g(X, PY).$$

The covariant derivative of F with respect to the Levi-Civita connection ∇^g is expressed in terms of dF and N in the following way (see e.g. [47])

$$(2.1) \quad \begin{aligned} 2(\nabla^g F)(X; Y, Z) &= -2g((\nabla_X^g P)Y, Z) = \\ &= dF(X, Y, Z) + dF(X, PY, PZ) + 4N(PX; Y, Z). \end{aligned}$$

The Lee form θ is defined by $\theta = \delta F \circ P$, where $\delta = - * d *$ is the co-differential with respect to g . For 1-form α we use the notation $P\alpha(X) = -\alpha(PX)$. Thus, $\theta = P\delta F$. We also have

$$\theta(X) = \sum_{i=1}^{2n} \epsilon_i (\nabla^g F)(e_i; e_i, PX) = \frac{1}{2} \sum_{i=1}^{2n} dF(e_i, Pe_i, X) = \sum_{i=1}^n dF(e_i, Pe_i, X).$$

Almost paraHermitian manifolds are classified with respect to the decomposition in invariant and irreducible subspaces, under the action of the structural group $U(n, \mathbb{A})$, of the vector space of tensors satisfying the same symmetries as $\nabla^g F$ [12, 30]. We recall the defining conditions of some of the classes:

- $\nabla^g F = 0 \Leftrightarrow dF = 0$, para-Kähler manifolds;
- $N = 0 \Leftrightarrow (\nabla_{PX}^g P)PY + (\nabla_X^g P)Y = 0$, paraHermitian manifolds [56];
- $(\nabla_X^g P)X = 0$, Nearly paraKähler manifolds;
- $dF = 0$, almost paraKähler manifolds;
- $dF = \theta \wedge F$, $d\theta = 0$, paraHermitian manifolds locally conformally equivalent to paraKähler spaces [30, 14].

Examples of almost paraHermitian manifolds including the non-compact hyperbolic Hopf and hyperbolic Calabi-Eckmann manifolds [13] are collected in [24]. Another source of examples comes from the k -symmetric spaces, i.e. homogeneous spaces defined by a Lie group automorphism of order k [10]. Almost paraHermitian manifolds are also called *almost bi-Lagrangian* [44, 48]. They arise in relation with the existence of Killing spinors of an indefinite neutral metric [48].

3. PARAHERMITIAN CONNECTIONS

A linear connection ∇ on an almost paraHermitian manifold (M, g, P) is said to be *paraHermitian connection*, if it preserves the paraHermitian structure, i.e. $\nabla g = \nabla P = 0$.

In this section we define canonical paraHermitian connections in a (formally) similar way as it was done in [33] for an almost Hermitian manifold.

We start with type decomposition of an element $B \in \Lambda^2(TM)$. Denote $g(X, B(Y, Z)) := B(X; Y, Z)$. Let $Bi(B) : \Lambda^2(TM) \rightarrow \Lambda^3$ be the Bianchi projector

$$3Bi(B)(X; Y, Z) = B(X; Y, Z) + B(Y; Z, X) + B(Z; X, Y).$$

Further, we say that B is

- of type (1,1) if $B(PX, PY) = -B(X, Y)$;
- of type (0,2) if $B(PX, Y) = -PB(X, Y)$;
- of type (2,0) if $B(PX, Y) = PB(X, Y)$.

We will denote the corresponding type-subspaces by $\Lambda^{1,1}$, $\Lambda^{0,2}$, $\Lambda^{2,0}$, respectively, such that $B = B^{1,1} \oplus B^{0,2} \oplus B^{2,0}$. The projections are given by

$$\begin{aligned} B^{1,1}(X, Y) &= \frac{1}{2} (B(X, Y) - B(PX, PY)), \\ B^{0,2}(X, Y) &= \frac{1}{4} (B(X, Y) + B(PX, PY) - PB(PX, Y) - PB(X, PY)), \\ B^{2,0}(X, Y) &= \frac{1}{4} (B(X, Y) + B(PX, PY) + PB(PX, Y) + PB(X, PY)) \end{aligned}$$

We define an involution $In : \Lambda^2(TM) \rightarrow \Lambda^2(TM)$ by $In(B)(X; Y, Z) = B(X; PY, PZ)$.

We may consider a 3-form ψ as a totally skew-symmetric section of $\Lambda^3(TM)$. It thus admits two different type decomposition:

1. decomposition as a 3-form: $\psi = \psi^+ \oplus \psi^-$, where ψ^+ denotes the $(1,2)+(2,1)$ -part and ψ^- -the $(3,0)+(0,3)$ -part of ψ given by

$$\begin{aligned} \psi^+(X, Y, Z) &= \frac{1}{4} (3\psi(X, Y, Z) - \psi(X, PY, PZ) - \psi(PX, Y, PZ) - \psi(PX, PY, Z)), \\ \psi^-(X, Y, Z) &= \frac{1}{4} (\psi(X, Y, Z) + \psi(X, PY, PZ) + \psi(PX, Y, PZ) + \psi(PX, PY, Z)). \end{aligned}$$

2. A type decomposition as an element of $\Lambda^2(TM)$.

The two decompositions are related by $\psi^- = \psi^{0,2}$, $\psi^+ = \psi^{2,0} + \psi^{1,1}$.

Let ∇ be any paraHermitian connection. Then we have

$$(3.2) \quad g(\nabla_X Y, Z) - g(\nabla_X^g Y, Z) = A(X; Y, Z),$$

where $A \in \Lambda^2(TM)$ since $\nabla g = 0$.

The torsion of ∇ , $T(X, Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X, Y]} \in \Lambda^2(TM)$ and

$$(3.3) \quad T = -A + 3Bi(A), \quad A = -T + \frac{3}{2}Bi(T), \quad Bi(A) = \frac{1}{2}Bi(T).$$

We determine ∇ in terms of its torsion.

Denote $d^a F(X, Y, Z) := -dF(PX, PY, PZ)$ we obtain easily the following

Proposition 3.1. *On an almost paraHermitian manifold we have:*

- a) *The Nijenhuis tensor is of type $(0,2)$. In particular it is trace-free, $tr(N) = 0$. The skew-symmetric part of N is given by*

$$(3.4) \quad Bi(N) = \frac{1}{3}(d^a F)^-;$$

- b) *The component $(\nabla^g F)^{1,1} = 0$.*
- c) *The component $(\nabla^g F)^{0,2}$ is determined by N :*

$$(3.5) \quad \begin{aligned} (\nabla^g F)^{0,2}(X; Y, Z) &= dF^-(X, Y, Z) + 2N(PX; Y, Z) = \\ &= N(PX; Y, Z) - N(PY; Z, X) - N(PZ; X, Y). \end{aligned}$$

- d) *The component $(\nabla^g F)^{2,0}$ is determined by dF^+ :*

$$(3.6) \quad (\nabla^g F)^{2,0}(X; Y, Z) = \frac{1}{2} (dF^+(X, Y, Z) + dF^+(X, PY, PZ))$$

We describe the paraHermitian connections in the next

Theorem 3.2. *Let ∇ be a paraHermitian connection. Then*

$$(3.7) \quad T^{0,2} = -N, \quad Bi(T^{2,0}) - Bi(T^{1,1}) = -\frac{1}{3}(d^a F)^+$$

For any 3-form ψ^+ of type $(1,2)+(2,1)$ and any section B_b of $\Lambda^{1,1}(TM)$ satisfying $Bi(B_b) = 0$ there exists a unique paraHermitian connection whose torsion T is given by the formula

$$(3.8) \quad T = -N - \frac{1}{8}(d^a F)^+ - \frac{3}{8}In(d^a F)^+ + \frac{9}{8}\psi^+ + \frac{3}{8}In(\psi^+) + B_b.$$

The corresponding paraHermitian connection is then equal to $\nabla^g + A$, where A is obtained from T by (3.3).

Proof. Since $\nabla P = 0$ we get the first equality in (3.7) by straightforward calculations. We calculate $T^{2,0} - T^{1,1} = N + In(T)$, $3Bi(In(T)) = -d^a F$. Apply (3.4) to derive $3(Bi(T^{2,0}) - Bi(T^{1,1})) = -d^a F + (d^a F)^- = -(d^a F)^+$ which completes the proof of (3.7).

Denote by ψ^+ the $(1,2)+(2,1)$ -form $Bi(T^{2,0}) + Bi(T^{1,1})$ and use (3.7) to get

$$(3.9) \quad Bi(T^{2,0}) = \frac{1}{2}\left(\psi^+ - \frac{1}{3}(d^a F)^+\right), \quad Bi(T^{1,1}) = \frac{1}{2}\left(\psi^+ + \frac{1}{3}(d^a F)^+\right).$$

A linear connection ∇ preserves the almost paracomplex structure if and only if A satisfies $A(X; PY, Z) + A(X; Y, PZ) = (\nabla^g F)(X; Y, Z)$. By means of (3.3) the last equality is equivalent to

$$(3.10) \quad -T(X; PY, Z) - T(X; Y, PZ) + \frac{3}{2}(Bi(T)(X; PY, Z) + Bi(T)(X; Y, PZ)) = (\nabla^g F)(X; Y, Z).$$

The first consequence of (3.9) and (3.10) is that the $(1,1)$ -part of T which satisfies the Bianchi identity is free, denote it by $T_b^{1,1} = B_b$. Take the $(0,2)$ and $(2,0)$ parts of (3.10), apply (3.5), (3.6) and use (3.7), (3.9) to get formula (3.8). \square

Corollary 3.3. *Let (M, g, P) be a $2n$ -dimensional almost paraHermitian manifold. There exists paraHermitian connection on M with totally skew-symmetric torsion if and only if the Nijenhuis tensor is totally skew-symmetric. In this case the connection is unique and the torsion T is given by*

$$(3.11) \quad T = (d^a F)^+ - N$$

Proof. Assume T is a 3-form. Then N is a 3-form due to (3.7) and $B_b = 0$. We claim $\psi^+ = (d^a F)^+$. Indeed, $\psi^+ = \frac{3}{4}T + \frac{1}{4}d^a F$. On the other hand, $\psi^+ = Bi(T^{2,0}) + Bi(T^{1,1}) = T + N = T + \frac{1}{3}(d^a F)^-$. Hence, the claim follows. Substituting $\psi^+ = (d^a F)^+$ into (3.8) we get (3.11). The corollary follows from Theorem 3.2 \square

We shall call this connection *the Bismut connection*.

Definition 3.4. A paraHermitian connection is called canonical if its torsion T satisfies the following conditions

$$(3.12) \quad T_b^{1,1} = 0, \quad (Bi(T))^+ = -\frac{2t-1}{3}(d^a F)^+$$

for some real parameter t . We denote the corresponding connection by ∇^t .

Combining (3.8) with (3.12) we get that the torsion T^t of ∇^t is given by

$$T^t = -N - \frac{3t-1}{4}(d^a F)^+ - \frac{t+1}{4}In(d^a F)^+.$$

Any canonical connection is connected with the Levi-Civita connection by

$$(3.13) \quad g(\nabla_X^t Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{2}g(\nabla_X^g P)(PY, Z) - \frac{t}{4}((d^a F)^+(X, Y, Z) - (d^a F)^+(X, PY, PZ)).$$

The paraHermitian connection with torsion 3-form is the canonical connection given by $t = -1$. Another remarkable connection is the canonical connection obtained for $t = 0$ [67],

$$g(\nabla_X^0 Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{2}g(\nabla_X^g P)(PY, Z), \quad T^0 = -N + \frac{1}{4}(d^a F)^+ - \frac{1}{4}In(d^a F)^+.$$

Note that if $dF^+ = 0$ then the real line of the canonical connections degenerates to a point ∇^0 with torsion $T^0 = -N$. Almost paraHermitian manifolds satisfying the condition $dF^+ = 0$ are called *quasi-paraKähler* or *(1,2)-symplectic*. In view of Proposition 3.1, quasi-paraKähler manifolds are characterized by [67], $(\nabla_{PX}^g F)(PY, Z) - ((\nabla_X^g F)(Y, Z) = 0$.

3.1. Canonical connection on paraHermitian manifold. We apply our previous discussion to a paraHermitian manifold, $N = 0$.

Theorem 3.5. *Let (M, g, P) be a $2n$ -dimensional paraHermitian manifold.*

- a) *There exists a unique paraHermitian connection ∇^1 on M with torsion $T^1 \in \Lambda^{2,0}(TM)$ i.e. T^1 satisfies*

$$T^1(PX, Y) = PT^1(X, Y).$$

This connection is the canonical connection obtained by $t = 1$ and given by

$$(3.14) \quad g(\nabla_X^1 Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{2}dF(PX, Y, Z).$$

- b) *The curvature $R^1 := [\nabla^1, \nabla^1] - \nabla_{[\cdot]}^1$ is of type $(1,1)$ in the sense that*

$$R^1(PX, PY) = -R^1(X, Y).$$

Proof. From $N = 0$ we get $(d^a F)^- = 0, d^a F = (d^a F)^+$. Apply Theorem 3.2. We have $B_b = 0, \psi^+ = Bi(T) = -\frac{1}{3}d^a F$ since $T^1 \in \Lambda^{2,0}(TM)$. Hence, this is the canonical connection obtained for $t = 1$ which proves a).

To prove b) we consider the paracomplex coordinate system $(x^1, \dots, x^n, \bar{x}^1, \dots, \bar{x}^n)$ around a point $p \in M$ such that $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ is an $+$ -eigen-basis of $T_p M^+$ and $\frac{\partial}{\partial \bar{x}^1}, \dots, \frac{\partial}{\partial \bar{x}^n}$ is an $-$ -eigen-basis of $T_p M^-$, i.e. $P\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$, $P\frac{\partial}{\partial \bar{x}^i} = -\frac{\partial}{\partial \bar{x}^i}$. Then the metric and the fundamental 2-form are given by $g = 2g_{i\bar{j}}dx^i d\bar{x}^j$, $F = F_{i\bar{j}}dx^i \wedge d\bar{x}^j$, $F_{i\bar{j}} = -F_{j\bar{i}} = -g_{i\bar{j}}$.

Summation in repeated indexes is always assumed. We use the following convention: For a tensor K of type (p, q) , the symbol $\overline{K_{i_1, \dots, i_q}^{j_1, \dots, j_p}}$ means $K_{i_1, \dots, i_q}^{\bar{j}_1, \dots, \bar{j}_p}$.

We derive easily the expressions

$$dF_{ijk} = dF_{i\bar{j}\bar{k}} = 0, \quad dF_{ij\bar{k}} = \frac{\partial g_{i\bar{k}}}{\partial x^j} - \frac{\partial g_{j\bar{k}}}{\partial x^i}, \quad dF_{i\bar{j}k} = \frac{\partial g_{k\bar{j}}}{\partial x^i} - \frac{\partial g_{k\bar{i}}}{\partial x^j} = -\overline{dF_{ij\bar{k}}}.$$

Due to the Koszul formula, the local components Γ_{ij}^k of the Levi-Civita connection are given by

$$(3.15) \quad \begin{aligned} \Gamma_{ij}^k &= \frac{1}{2}g^{k\bar{s}} \left(\frac{\partial g_{i\bar{s}}}{\partial x^j} + \frac{\partial g_{j\bar{s}}}{\partial x^i} \right), \quad \Gamma_{i\bar{j}}^{\bar{k}} = \frac{1}{2}g^{\bar{k}s} \left(\frac{\partial g_{is}}{\partial x^{\bar{j}}} + \frac{\partial g_{\bar{j}s}}{\partial x^{\bar{i}}} \right), \\ \Gamma_{i\bar{j}}^k &= \frac{1}{2}g^{k\bar{s}} dF_{\bar{j}s i} = \Gamma_{ji}^k, \quad \Gamma_{i\bar{j}}^{\bar{k}} = \frac{1}{2}g^{s\bar{k}} dF_{s i \bar{j}} = \Gamma_{\bar{j}i}^{\bar{k}}, \quad \Gamma_{ij}^{\bar{k}} = \Gamma_{i\bar{j}}^k = 0. \end{aligned}$$

The local components C_{ij}^k of ∇^1 are calculated from (3.14) and (3.15)

$$(3.16) \quad C_{ij}^k = g^{k\bar{s}} \frac{\partial g_{j\bar{s}}}{\partial x^i}, \quad C_{i\bar{j}}^{\bar{k}} = g^{s\bar{k}} \frac{\partial g_{s\bar{j}}}{\partial x^{\bar{i}}}, \quad C_{ij}^k = C_{i\bar{j}}^k = C_{ij}^{\bar{k}} = C_{i\bar{j}}^{\bar{k}} = C_{ij}^{\bar{k}} = C_{i\bar{j}}^{\bar{k}} = 0$$

The curvature tensor R^1 has the property $R^1 \circ P = P \circ R^1$ since $\nabla^1 P = 0$. To prove b) it is sufficient to show $R_{ijkl}^1 = R_{i\bar{j}\bar{k}l}^1 = 0$ which is a direct consequence of (3.16). \square

Further we shall call ∇^1 *the Chern connection*. This connection coincides with the canonical compatible connection of the tangent bundle viewed as a paraHermitian, paraholomorphic bundle of rank n defined in [28].

Corollary 3.6. *The curvature R^g of the Levi-Civita connection of a paraHermitian manifold satisfies the identities*

$$R_{ijkl}^g = R_{i\bar{j}\bar{k}l}^g = 0,$$

equivalently

$$\begin{aligned} R^g(X, Y, Z, V) + R^g(PX, PY, PZ, PV) + R^g(X, Y, PZ, PV) + R^g(X, PY, Z, PV) + \\ R^g(X, PY, PZ, V) + R^g(PX, PY, Z, V) + R^g(PX, Y, PZ, V) + R^g(PX, Y, Z, PV) = 0. \end{aligned}$$

The curvature R^1 and the torsion T^1 of the Chern connection are given by

$$R_{ijkl}^1 = -g_{s\bar{l}} \frac{\partial C_{ik}^s}{\partial x^{\bar{j}}} = -g_{s\bar{l}} \frac{\partial}{\partial x^{\bar{j}}} \left(g^{s\bar{m}} \frac{\partial g_{k\bar{m}}}{\partial x^i} \right), \quad T_{\bar{k}ij} = dF_{ij\bar{k}}.$$

3.2. Ricci forms of the canonical connections. For a linear connection ∇ with curvature tensor R on an almost paraHermitian manifold of dimension $2n$ we have Ricci type tensors:

- the Ricci tensor $\rho(X, Y) := \sum_{i=1}^{2n} \epsilon_i R(e_i, X, Y, e_i)$;
- the *-Ricci tensor $\rho^*(X, Y) := \sum_{i=1}^{2n} \epsilon_i R(e_i, X, PY, Pe_i)$;
- the Ricci form $r(X, Y) := -\frac{1}{2} \sum_{i=1}^{2n} \epsilon_i R(X, Y, e_i, Pe_i) = -\sum_{i=1}^n R(X, Y, e_i, Pe_i)$.

The scalar curvatures are defined to be the corresponding trace:

- the scalar curvature $s = \text{tr}_g \rho = \sum_{i=1}^{2n} \epsilon_i \rho(e_i, e_i)$,
- the *-scalar curvature $s^* = \text{tr}_g \rho^* = 2 \sum_{i=1}^{2n} \epsilon_i \rho(e_i, e_i)$,
- the trace of the Ricci form $\tau = \text{tr}_g r = \sum_{i,j=1}^n r(e_i, Pe_j)$.

For the Levi-Civita connection, we have the properties (see [56])

$$\rho^{g*}(X, Y) = r^g(X, PY), \quad \rho^{g*}(X, Y) + \rho^{g*}(PY, PX) = 0 \text{ and consequently, } s^{g*} = \tau^g.$$

To find relations between the Ricci forms of the canonical Hermitian connection we consider the paraholomorphic canonical bundle $\Lambda_{\mathbb{A}}^n(TM)$. Any linear connection preserving the structure P , i.e. preserving the eigensubbundles TM^+ and TM^- , induces a connection on the line bundle $\Lambda_{\mathbb{A}}^n(TM)$ with curvature equal to (-) its Ricci form. Let s be a section of $\Lambda_{\mathbb{A}}^n(TM)$. From (3.13) we infer that $\nabla^t s = \nabla^0 s + \frac{t}{2} P\theta \otimes s$. Consequently $r^t = r^0 - \frac{t}{2} d(P\theta)$. In particular the Ricci forms of the Bismut and Chern connection are related by

$$(3.17) \quad r^{-1} = r^1 + d(P\theta).$$

4. PARAHERMITIAN 4-MANIFOLD

In this section we find a paraHermitian analogue of the Apostolov-Gauduchon generalization [6] of the Goldberg-Sachs theorem in General Relativity (see e.g. [57]). We prove the result using the properties of the Chern connection.

Let (M, g) be an oriented pseudo-Riemannian 4-manifold with neutral metric g of signature $(+, +, -, -)$. This is equivalent, on one hand to the existing of an almost paracomplex structure, and on the other hand, to the existence of two kinds of almost complex structures. In a compact case the second property leads to topological obstruction to the existence of neutral metric expressed in terms of the signature and the Euler characteristic [53].

The bundle $\Lambda^2 M$ of real 2-forms of a neutral Riemannian 4-manifold splits

$$(4.18) \quad \Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M,$$

where $\Lambda^+ M$, resp. $\Lambda^- M$ is the bundle of self-dual, resp. anti-self-dual 2-forms, i.e. the eigen-sub-bundle with respect to the eigenvalue $+1$, resp. -1 , of the Hodge $*$ -operator acting as an involution on $\Lambda^2 M$. We also may consider the connected component $SO^+(2, 2)$ of the structure group $SO(2, 2)$. This group has the splitting $SO^+(2, 2) = SL(2) \times SL(2)$ which defines two real vector bundles (of rank 2) S^+ and S^- and $TM = S^+ \oplus S^-$ which induces the splitting (4.18).

We will freely identify vectors and co-vectors via the metric g .

The self-dual part $W^+ = \frac{1}{2}(W + *W)$ of the Weyl tensor W is viewed as a section of the bundle $\mathbb{W}^+ = Sym_0 \Lambda^+ M$ of symmetric traceless endomorphisms of $\Lambda^+ M$.

Let P be an almost paracomplex structure compatible with the metric g such that (g, P) defines an almost paraHermitian structure. Then the fundamental 2-form F is a section of $\Lambda^+ M$ and has constant norm 2. Conversely, any smooth section of $\Lambda^+ M$ with constant norm 2 is the fundamental 2-form of an almost paracomplex structure. Our considerations in this section are complementary to that in [5] in the sense that a section of $\Lambda^+ M$ with norm -2 can be considered as a Kähler form of an almost complex structure, the case investigated in [5].

We have the following orthogonal splitting for $\Lambda^+ M$

$$(4.19) \quad \Lambda^+ M = \mathbb{R} \cdot F \oplus \Lambda_0^+ M,$$

where $\Lambda_0^+ = \Lambda^{0,2} M \oplus \Lambda^{2,0} M$ denotes the bundle of P -invariant real 2-forms ϕ , $\phi(PX, PY) = \phi(X, Y)$.

In accordance with (4.19) the bundle \mathbb{W}^+ splits into three pieces as follows:

$$\mathbb{W}^+ = \mathbb{W}_1^+ \oplus \mathbb{W}_2^+ \oplus \mathbb{W}_3^+,$$

where

- $\mathbb{W}_1^+ = \mathbb{M} \times \mathbb{R}$ is the sub-bundle of elements preserving (4.19) and acting by the homothety on the two factors, hence the trivial line bundle generating by the elements $\frac{3}{4}F \otimes F - \frac{1}{2}id$;
- \mathbb{W}_2^+ is the sub-bundle of elements which exchange the two factors in (4.19): each element $\phi \in \Lambda_0^+ M$ is identified with the element $\frac{1}{2}(F \otimes \phi + \phi \otimes F)$;
- \mathbb{W}_3^+ is the subbundle of elements preserving the splitting (4.19) and acts trivially on the first factor $\mathbb{R} \cdot F$, i.e. it is the space of those endomorphisms of Λ_0^+ which are P -invariant.

Thus, W^+ can be written in the form

$$(4.20) \quad W^+ = f \left(\frac{3}{4}F \otimes F - \frac{1}{2}id \right) + \frac{1}{2}(F \otimes \phi + \phi \otimes F) + W_3^+,$$

where f is some real function.

In dimension 4 the Lee form θ determines dF completely by

$$(4.21) \quad dF = \theta \wedge F.$$

In particular, $d\theta$ is trace-free, $\sum_{i=1}^2 d\theta(e_i, Pe_i) = 0$. Hence, the self-dual part $d\theta^+$ of $d\theta$ is a section of $\Lambda_0^+ M$.

To any 4-dimensional almost paraHermitian manifold with a Lee form θ one can associate the canonical Weyl structure, i.e. a torsion-free connection ∇^w determined by the equation $\nabla^w g = \theta \otimes g$. The conformal scalar curvature k of an almost paraHermitian structure is defined to be the scalar curvature of the canonical Weyl structure with respect to the metric g . Then (see e.g. [32])

$$(4.22) \quad k = s - \frac{3}{2} (g(\theta, \theta) + 2\delta\theta).$$

The conformal scalar curvature is conformally invariant of weight -2 , i.e. if $g' = f^{-2}g$ then $k' = f^2k$.

4.1. Curvature of paraHermitian 4-manifold. Let (M, g, P) be a 4-dimensional paraHermitian manifold. The Chern connection ∇^1 and the Levi-Civita connection are related by $g(\nabla_X^1 Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{2}(\theta \wedge F)(PX, Y, Z)$ due to (3.14) and (4.21). Consequently,

$$(4.23) \quad R^1(X, Y, Z, V) = R^g(X, Y, Z, V) - \frac{1}{2}d(P\theta)(X, Y)F(V, Z) + \frac{1}{2}(L(Y, Z)g(V, X) - L(X, Z)g(V, Y) + L(X, V)g(Y, Z) - L(Y, V)g(Z, X)),$$

where the tensor L has the form

$$(4.24) \quad L(X, Y) = (\nabla_X^g \theta)Y + \frac{1}{2}\theta(X)\theta(Y) - \frac{1}{4}g(\theta, \theta)g(X, Y).$$

The curvature R^1 is of type (1,1) according to Theorem 3.5. Then (4.23), (4.24) imply, in local paraholomorphic coordinates, that

$$(4.25) \quad \begin{aligned} R_{ijk\bar{l}}^g &= -\frac{1}{2}(L_{jk}g_{i\bar{l}} - L_{ik}g_{j\bar{l}} + d\theta_{ij}g_{k\bar{l}}), \quad R_{i\bar{j}k\bar{l}}^g = \overline{R_{ijk\bar{l}}^g}; \\ R_{ij\bar{k}\bar{l}}^g &= -\frac{1}{2}(L_{j\bar{k}}g_{i\bar{l}} - L_{i\bar{k}}g_{j\bar{l}} - L_{j\bar{l}}g_{i\bar{k}} + L_{i\bar{l}}g_{j\bar{k}}), \quad R_{i\bar{j}k\bar{l}}^g = \overline{R_{ij\bar{k}\bar{l}}^g}, \\ L_{ij} &= \nabla_i^g \theta_j + \frac{1}{2}\theta_i \theta_j, \quad L_{i\bar{j}} = \nabla_i^g \theta_{\bar{j}} + \frac{1}{2}\theta_i \theta_{\bar{j}} - \frac{1}{4}|\theta|^2 g_{i\bar{j}}. \end{aligned}$$

We take the traces in (4.23), (4.25), (3.17) and use (4.22) to get our technical

Proposition 4.1. *The Ricci tensors and the scalar curvatures of a 4-dimensional paraHermitian manifold satisfy the conditions*

$$\begin{aligned} \rho_{jk}^g &= R_{ijk\bar{i}}^g + R_{ikj\bar{i}}^g = -\frac{1}{2}(\nabla_j^g \theta_k + \nabla_k^g \theta_j + \theta_j \theta_k), \quad \rho_{j\bar{k}}^g = \overline{\rho_{jk}^g}, \\ \rho_{jk}^{g*} &= -R_{ijk\bar{i}}^g + R_{ikj\bar{i}}^g = -\frac{1}{2}d\theta_{jk}, \quad \rho_{j\bar{k}}^{g*} = \overline{\rho_{jk}^{g*}}, \\ \rho_{j\bar{k}}^g + \rho_{j\bar{k}}^{g*} &= 2R_{ijk\bar{i}}^g = \left(\frac{1}{2}\delta\theta + \frac{1}{4}g(\theta, \theta)\right)g_{j\bar{k}}, \\ s + s^* &= 2\delta\theta + g(\theta, \theta), \\ k &= -\frac{1}{2}(s + 3s^*) = -\tau^{-1}. \end{aligned}$$

In particular, the conformal scalar curvature is equal to $(-)$ the trace of the Ricci form of the Bismut connection. Therefore, the trace of the Ricci form of the Bismut connection is a conformal invariant of weight -2 .

We note that the expression of the $(1,1)$ -part of the sum of the two Ricci tensors and the formula for the sum of the two scalar curvatures in Proposition 4.1 were obtained in [56].

The structure of W^+ on a 4-dimensional paraHermitian manifold is similar to that of the Hermitian manifold presented in [6]. We described it in the following

Lemma 4.2. *On a 4-dimensional paraHermitian manifold the third component W_3^+ of W^+ vanishes identically and the positive Weyl tensor is given by*

$$(4.26) \quad W^+ = \frac{k}{8}F \otimes F - \frac{k}{12}id - \frac{1}{4}\psi \otimes F - \frac{1}{4}F \otimes \psi,$$

where the two form ψ is determined by the self-dual part of $d\theta_+$, $\psi_{ij} = d\theta_{ij}$.

Proof. On a 4-dimensional pseudo-Riemannian manifold the Weyl tensor is expressed in terms of the normalized Ricci tensor $h = -\frac{1}{2}(\rho - \frac{s}{6}g)$ as follows

$$(4.27) \quad \begin{aligned} W(X, Y, Z, V) = & R^g(X, Y, Z, V) - h(X, Z)g(Y, V) + \\ & h(Y, Z)g(X, V) - h(Y, V)g(X, Z) + h(X, V)g(Y, Z). \end{aligned}$$

The condition $W_3^+ = 0$ is a consequence of (4.27) and Corollary 3.6, due to the relation $W_{ijkl} = R_{ijkl}^g = 0$. According to (4.20) we have $W(F) = W^+(F) = fF + \psi$. We calculate from (4.27) applying Proposition 4.1 that

$$W_{ijk\bar{k}} = -\frac{1}{2}d\theta_{ij}, \quad W_{i\bar{j}k\bar{k}} = \frac{k}{6}g_{i\bar{j}}.$$

Hence, the lemma follows. \square

Another glance at (3.17) leads to the expression $r_{ij}^{-1} = -d\theta_{ij}$, $r_{i\bar{j}}^{-1} = d\theta_{i\bar{j}}$ since the Ricci form of the Chern connection is of type $(1,1)$. The last equalities and Lemma 4.2 imply

Proposition 4.3. *A 4-dimensional paraHermitian manifold is anti-self-dual ($W^+ = 0$) if and only if the Ricci form of the Bismut connection is an anti-self-dual 2-form.*

Consider the co-differential of the positive Weyl tensor δW^+ as an element of $\Lambda_0^2(T^*M)$. Then we have the splitting

$$\delta W^+ = (\delta W^+)^+ \oplus (\delta W^+)^-,$$

where $(\delta W^+)^+$ is a section of $\Lambda_0^{(2,0)+(1,1)}(T^*M)$ while $(\delta W^+)^-$ is a section of $\Lambda_0^{0,2}(T^*M)$. In particular, $(\delta W^+)^- = 0$ if and only if the co-differential of the whole Weyl tensor vanishes on any three $(1,0)$ -vectors.

4.2. Proof of Theorem 1.1. The equivalence a) \Leftrightarrow b) is proved in Lemma 4.2.

The second Bianchi identity reads as

$$\delta W(X; Y, Z) = (\nabla_Y^g h)(Z, X) - (\nabla_Z^g h)(Y, X)$$

On $(1,0)$ vectors it gives due to (3.15) that

$$(4.28) \quad (\delta W^+)(X^{1,0}; Y^{1,0}, Z^{1,0}) = (\nabla_{Y^{1,0}}^g \rho)(Z^{1,0}, X^{1,0}) - (\nabla_{Z^{1,0}}^g \rho)(Y^{1,0}, X^{1,0}).$$

Assume $d\theta_{ij} = 0$. Then Proposition 4.1, the Ricci identities and (4.25) imply $\nabla_i^g \rho_{jk} - \nabla_j^g \rho_{ik} = 0$. Hence, $(\delta W^+)^- = 0$ due to (4.28). The implication a) \Rightarrow c) is proved.

Let $(\delta W^+)^- = 0$. The local components of the Chern connection and its torsion tensor are given by

$$C_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}(\theta_i \delta_j^k - \theta_j \delta_i^k), \quad T_{ij}^k = \theta_i \delta_j^k - \theta_j \delta_i^k.$$

The equation (4.28), in terms of the Chern connection, takes the form

$$(4.29) \quad \nabla_i^1 \rho_{jk} - \nabla_j^1 \rho_{ik} = \frac{3}{2}(\theta_j \rho_{ik} - \theta_i \rho_{jk}).$$

The Ricci identities for the Chern connection, $\nabla_i^1 \nabla_j^1 \theta_k - \nabla_j^1 \nabla_i^1 \theta_k = \theta_j \nabla_i^1 \theta_k - \theta_i \nabla_j^1 \theta_k$, the first equality in Proposition 4.1 and (4.29) yield

$$\nabla_i^1 d\theta_{jk} - \nabla_j^1 d\theta_{ik} = \theta_k d\theta_{ij} - \frac{3}{2}(\theta_i d\theta_{jk} - \theta_j d\theta_{ik}).$$

Make a cyclic permutation in the latter then add the two and subtract the third of the obtained equalities to get

$$(4.30) \quad \nabla_i^1 d\theta_{jk} = -2\theta_i d\theta_{jk} + \frac{1}{2}(\theta_j d\theta_{ki} - \theta_k d\theta_{ji}).$$

Take the covariant derivative in (4.30) and apply (4.30) to the obtained result to derive

$$(4.31) \quad \begin{aligned} \nabla_l^1 \nabla_i^1 d\theta_{jk} &= -2\nabla_l^1 \theta_i d\theta_{jk} + \frac{1}{2}(\nabla_l^1 \theta_j d\theta_{ki} - \nabla_l^1 \theta_k d\theta_{ji}) \\ &\quad + 4\theta_i \theta_l d\theta_{jk} + \frac{5}{4}(\theta_i \theta_j d\theta_{lk} - \theta_i \theta_k d\theta_{lj}) - (\theta_j \theta_l d\theta_{ki} - \theta_k \theta_l d\theta_{ji}) \end{aligned}$$

The Ricci identity $\nabla_i^1 \nabla_j^1 d\theta_{kl} - \nabla_j^1 \nabla_i^1 d\theta_{kl} = -\theta_i \nabla_j^1 d\theta_{kl} + \theta_j \nabla_i^1 d\theta_{kl}$ and (4.31) imply

$$\begin{aligned} 2d\theta_{li} d\theta_{jk} + \frac{3}{4}(\theta_i \theta_j d\theta_{kl} + \theta_i \theta_k d\theta_{lj} - \theta_j \theta_l d\theta_{ki} - \theta_k \theta_l d\theta_{ij}) = \\ \frac{1}{2}(d\theta_{ij} \nabla_l^1 \theta_k + d\theta_{ki} \nabla_l^1 \theta_j - d\theta_{kl} \nabla_i^1 \theta_j - d\theta_{lj} \nabla_i^1 \theta_k). \end{aligned}$$

Change $l \leftrightarrow j$, $i \leftrightarrow k$ into the latter equality and sum up the results to obtain

$$4d\theta_{li} d\theta_{jk} = d\theta_{lk} d\theta_{ij} + d\theta_{lj} d\theta_{ki}.$$

From the last equality we easily infer $5d\theta_{li} d\theta_{jk} = 0$. Hence, $d\theta_{jk} = 0$ which completes the proof of Theorem 1.1. \square

We consider the question of integrability of totally isotropic real 2-plane supplementary distributions on an oriented 4-dimensional neutral Riemannian manifold. Any such splitting of the tangent bundle defines an almost paracomplex structure compatible with the neutral metric, such that we get an almost paraHermitian 4-manifold. The integrability of 2-plane supplementary distributions is equivalent to the integrability of the almost paracomplex structure. A necessary condition is the vanishing of the third component W_3^+ of the positive Weyl tensor, which is equivalent to the vanishing of the whole Weyl tensor on the 2-plane distribution. Note that this is equivalent to the vanishing of the whole curvature on the 2-plane i.e. the identity in Corollary 3.6 holds. This leads to fourth order polynomial equation [1] (see also [5]) which can not have always real-root solutions. In the case of existence, we give sufficient conditions for the integrability of P in the following

Theorem 4.4. *Let (M, g) be an oriented neutral Riemannian 4-manifold with nowhere vanishing positive Weyl tensor W^+ . Suppose that P is an almost paracomplex structure such that W^+ vanishes on each eigen-subbundle determined by P , i.e. the component W_3^+ of W^+ with respect to P vanishes. Then any of the two following three conditions imply the third:*

- i) $W_2^+ = 0$;
- ii) $(\delta W^+)^- = 0$;
- iii) the paracomplex structure P is integrable.

Proof. Observe that any smooth section F of $\Lambda^+ M$ with constant norm 2 is the fundamental 2-form of an almost paracomplex structure. Replacing M by a two-fold covering, if necessary, the positive Weyl tensor W^+ can be written in the form (4.20), where f is a smooth function and $W_3^+ = 0$.

According to Theorem 1.1 we have to show that i) and ii) imply iii).

Assume $W_2^+ = 0$. Then $W^+ = \frac{3}{4}fF \otimes F - \frac{1}{2}fid$. Using the definition of the Lee form, we calculate easily that

$$(4.32) \quad (\delta W^+)_X = \left(\frac{1}{2}Pdf(X) - \frac{3}{4}fP\theta(X) \right) F - \frac{3}{4}f\nabla_{PX}^g F + \frac{1}{4}(df \wedge X + Pdf \wedge PX).$$

The (0,2)-part of (4.32) gives $0 = (\delta W^+)^- = (\nabla^g F)^{0,2}$. Using (3.5) we infer $N = 0$. \square

Corollary 4.5. *Let (M, g, P) be an almost paraHermitian 4-manifold.*

- i) *Suppose $W^+ \neq 0$ everywhere and $W_2^+ = W_3^+ = 0$. Then $(\delta W^+)^+ = 0$ is equivalent to $d(|W^+|^{-\frac{2}{3}}F) = 0$.*
- ii) *Suppose (M, g, P) is a paraHermitian 4-manifold. If it has nowhere vanishing positive Weyl tensor then $\delta W^+ = 0$ if and only if $g' = |W^+|^{-\frac{2}{3}}g$ is a paraKähler metric. The Ricci tensor ρ^g of g is P -anti-invariant if and only if the vector field $P\text{grad}_{g'}f$, where $f = |W^+|^{-\frac{1}{3}}$ is a Killing vector field with respect to the paraKähler metric g' .*

In particular, a paraHermitian Einstein 4-manifold is either with everywhere vanishing positive Weyl tensor or is globally conformal to a paraKähler space. In the latter case there exists non zero Killing vector field with respect to the paraKähler metric.

Proof. The (2,0)+(1,1)-part of (4.32) yields

$$\begin{aligned} (\delta W^+)_X^+ &= \frac{3}{2} \left(\frac{1}{3}Pdf(X) - \frac{1}{2}fP\theta(X) \right) F + \\ &+ \frac{3}{4} \left[\left(\frac{1}{3}df - \frac{1}{2}f\theta \right) \wedge X + \left(\frac{1}{3}Pdf - \frac{1}{2}fP\theta \right) \wedge PX \right]. \end{aligned}$$

Assume $W_2^+ = 0$. Then the function f is nowhere vanishing otherwise W^+ will have zeros. Moreover $|W^+|^2 = (W^+(F, F))^2 = 4f^2$. The equation $(\delta W^+)^+ = 0$ is equivalent to $\theta = \frac{2}{3}d\ln f$. Thus we prove i). The condition ii) is a consequence of i) and Theorem 4.4. \square

5. NEARLY PARAKÄHLER MANIFOLDS

An almost paraHermitian manifold is called *Nearly paraKähler (nearly bi-Lagrangian)* if the almost paraHermitian structure is not para-Kähler and satisfies the identity

$$(\nabla_X^g P)X = 0, \quad \Leftrightarrow (\nabla_X^g F)(Y, Z) + (\nabla_Y^g F)(X, Z) = 0.$$

An example of nearly paraKählerian 6-manifold is given in [13].

We denote the unique canonical connection ∇^0 on a Nearly paraKähler manifold by ∇ .

Applying the statements in Proposition 3.1, we get

Proposition 5.1. *A nearly paraKähler manifold is quasi-Kähler, $dF^+ = 0$, the Nijenhuis tensor N is a 3-form and the torsion T of the unique canonical connection is determined by the Nijenhuis tensor, $T = -N = P\nabla P$.*

Many properties of nearly paraKähler manifolds are, in some sense, formally very similar to these of nearly Kähler manifolds studied mainly by A.Gray [35, 36, 37]. Below we follow roughly [37, 50] (see also [15]).

Proposition 5.2. *On a nearly paraKähler manifold the following identity holds*

$$(5.33) \quad R^g(X, Y, Z, V) + R^g(X, Y, PZ, PV) = g((\nabla_X^g P)Y, (\nabla_Z^g P)V).$$

Proof. The nearly paraKähler condition implies $(\nabla_X^g P)(Y, PY) = 0$. Then, we get easily that $R^g(X, Y, X, Y) + R^g(X, Y, PX, PY) = g((\nabla_X^g P)Y, (\nabla_X^g P)Y)$. Polarizing the latter equality and using Bianchi identity, we obtain (5.33). \square

Our crucial result in this section is the following

Theorem 5.3. *On a nearly paraKähler manifold the Nijenhuis tensor is parallel with respect to the canonical connection ∇ ,*

$$\nabla N = -\nabla T = 0.$$

Proof. The curvature R^g of the Levi-Civita connection and the curvature R of the canonical connection are related by

$$(5.34) \quad \begin{aligned} R^g(X, Y, Z, V) &= R(X, Y, Z, V) - \frac{1}{2}(\nabla_X T)(Y, Z, V) + \frac{1}{2}(\nabla_Y T)(X, Z, V) \\ &\quad - \frac{1}{2}g(T(X, Y), T(Z, V)) - \frac{1}{4}g(T(Y, Z), T(X, V)) - \frac{1}{4}g(T(Z, X), T(Y, V)) \end{aligned}$$

Since $R \circ P = P \circ R$, the equation (5.34) leads to

$$\begin{aligned} R^g(X, Y, Z, V) + R^g(X, Y, PZ, PV) &= \\ &= -(\nabla_X T)(Y, Z, V) + (\nabla_Y T)(X, Z, V) - g(T(X, Y), T(Z, V)). \end{aligned}$$

Comparing the latter equality with (5.33), we derive $(\nabla_X T)(Y, Z, V) - (\nabla_Y T)(X, Z, V) = 0$. Take the cyclic sum and add the result to conclude $\nabla T = 0$. \square

Corollary 5.4. *On a nearly paraKähler manifold the following identities hold*

$$R^g(X, Y, Z, V) + R^g(X, Y, PZ, PV) + R^g(PX, Y, PZ, V) + R^g(PX, Y, Z, PV) = 0,$$

$$R^g(X, Y, Z, V) = R^g(PX, PY, PZ, PV);$$

$$R(X, Y, Z, V) = R(Z, V, X, Y) = -R(PX, PY, Z, V) = -R(X, Y, PZ, PV);$$

$$\rho^g(PX, PY) = -\rho^g(X, Y), \quad \rho^{g*}(PX, PY) = -\rho^{g*}(X, Y) = -\rho^{g*}(Y, X),$$

$$\rho(X, Y) = \rho(Y, X), \quad r(PX, PY) = -r(X, Y),$$

$$(5.35) \quad \rho^g(X, Y) - \rho(X, Y) = \frac{1}{2} \sum_{i=1}^n g(T(X, e_i), T(Y, e_i)),$$

$$(5.36) \quad \rho^g(X, Y) + \rho^{g*}(X, Y) = 2 \sum_{i=1}^n g(T(X, e_i), T(Y, e_i)),$$

$$(5.37) \quad \rho^{g*}(X, Y) = r^g(X, PY) = r(X, PY) - \frac{1}{2} \sum_{i=1}^n g(T(X, e_i), T(Y, e_i)),$$

$$(5.38) \quad 3\rho^g(X, Y) - \rho^{g*}(X, Y) = 4\rho(X, Y),$$

$$(5.39) \quad \rho^g(X, Y) + 5\rho^{g*}(X, Y) = 4r(X, PY).$$

Proof. Put $\nabla T = 0$ into (5.34) to get

$$(5.40) \quad \begin{aligned} R^g(X, Y, Z, V) &= R(X, Y, Z, V) \\ &- \frac{1}{2}g(T(X, Y), T(Z, V)) - \frac{1}{4}g(T(Y, Z), T(X, V)) - \frac{1}{4}g(T(Z, X), T(Y, V)) \end{aligned}$$

All the identities in the corollary are easy consequences of (5.40) \square

5.1. Nearly paraKähler manifolds of dimension 6. We recall that a nearly paraKähler manifold is said to be of *constant type* $\alpha \in \mathbb{R}$ if

$$(5.41) \quad g((\nabla_X^g P)Y, (\nabla_X^g P)Y) = \alpha (g(X, X)g(Y, Y) - g^2(X, Y) + g^2(PX, Y)).$$

In the Nearly Kähler case the constant type condition (with positive constant α) occurs only in dimension 6 and any 6-dimensional Nearly Kähler manifold is an Einstein manifold with positive scalar curvature [37]. It is observed in [48] that in the Nearly paraKähler case the constant type phenomena occurs but the zero-value of α can not be excluded. We describe the structure of the Ricci tensor in the next

Theorem 5.5. *Any 6-dimensional nearly paraKähler manifold is an Einstein manifold of constant type $\alpha \in \mathbb{R}$ and the following relations hold*

$$(5.42) \quad \rho^g = 5\alpha g, \quad \rho^{g*} = -\alpha g, \quad \rho = 4\alpha g.$$

Consequently, the Riemannian scalar curvature $s^g = 30\alpha$.

In particular, if $\alpha = 0$ then the manifold is Ricci flat.

Proof. Let $e_1, e_2, e_3, Pe_1, Pe_2, Pe_3$ be an orthonormal local basis of smooth vector fields. The torsion T of the canonical connection (or equivalently, the Nijenhuis tensor N) is a 3-form of type $(3,0)+(0,3)$. Therefore we may write $T(e_1, e_2) = ae_3 + bPe_3$, where a and b are smooth functions which turn to be constants because the torsion is ∇ -parallel. It is easy to calculate that (5.41) holds with $\alpha = a^2 - b^2$. Moreover, we get the formula

$$(5.43) \quad \sum_{i=1}^3 g(T(X, e_i), T(Y, e_i)) = 2(a^2 - b^2)g(X, Y) = 2\alpha g(X, Y).$$

The Nearly paraKähler condition implies $(a, b) \neq (0, 0)$. In particular, the $(3,0)+(0,3)$ -form T is non-degenerate. On the other hand, $\nabla T = 0$, due to Theorem 5.3. Hence, the Ricci 2-form of the canonical connection vanishes as a curvature of a flat line bundle. The condition $r = 0$ and Corollary 5.4 completes the proof. \square

Remark 5.6. Using similar arguments as in the Nearly Kähler situation [37] we derive that a Nearly paraKähler manifold of non-zero constant type has to be of dimension 6.

6. EXAMPLES, TWISTORS AND REFLECTORS ON PARAQUATERNIONIC MANIFOLDS

To obtain examples of Nearly paraKähler manifolds we involve twistor machinery. We are going to adapt Salamon's twistor construction on quaternionic manifolds [60, 61, 62] to the paraquaternionic spaces.

6.1. Paraquaternionic manifolds. Both quaternions H and paraquaternions \tilde{H} are real Clifford algebras, $H = C(2, 0)$, $\tilde{H} = C(1, 1) \cong C(0, 2)$. In other words, the algebra \tilde{H} of paraquaternions is generated by the unity 1 and the generators J_1, J_2, J_3 satisfying the *paraquaternionic identities*,

$$(6.44) \quad J_1^2 = J_2^2 = -J_3^2 = 1, \quad J_1 J_2 = -J_2 J_1 = J_3.$$

We recall the notion of almost paraquaternionic manifold introduced by Libermann [51]. An *almost quaternionic structure of the second kind* on a smooth manifold consists of two almost product structures J_1, J_2 and an almost complex structure J_3 , which mutually anti-commute, i.e. these structures satisfy the paraquaternionic identities (6.44). Such a structure is also called *complex product structure* [4, 2].

An *almost hyper-paracomplex structure* on a $4n$ -dimensional manifold M is a triple $\tilde{H} = (J_\alpha), \alpha = 1, 2, 3$, where $J_\alpha, \alpha = 1, 2$ are almost paracomplex structures $J_\alpha : TM \rightarrow TM$, and $J_3 : TM \rightarrow TM$ is an almost complex structure, satisfying the paraquaternionic identities (6.44). When each $J_\alpha, \alpha = 1, 2, 3$ is an integrable structure, \tilde{H} is said to be a *hyper-paracomplex structure* on M . Such a structure is also called sometimes *pseudo-hyper-complex* [26]. Any hyper-paracomplex structure admits a unique torsion-free connection ∇^{ob} preserving J_1, J_2, J_3 [4, 2] called *the complex product connection*.

In fact an almost hyper-paracomplex structure is hyper-paracomplex if and only if any two of the three structures J_α are integrable, due to the following

Proposition 6.1. *The Nijenhuis tensors N_α of an almost hyper-paracomplex structure $\tilde{H} = (J_\alpha), \alpha = 1, 2, 3$ are related by:*

$$\begin{aligned} 2N_\alpha(X, Y) &= N_\beta(J_\gamma X, J_\gamma Y) - J_\gamma N_\beta(J_\gamma X, Y) - J_\gamma N_\beta(X, J_\gamma Y) - J_\gamma^2 N_\beta(X, Y) + \\ &N_\gamma(J_\beta X, J_\beta Y) - J_\beta N_\gamma(J_\beta X, Y) - J_\beta N_\gamma(X, J_\beta Y) - J_\beta^2 N_\gamma(X, Y) \end{aligned}$$

Proof. The formula follows by very definitions with long but standard computations. \square

We note that during the preparation of the manuscript the formula in the Proposition 6.1 appeared in the context of Lie algebras in [22].

An *almost paraquaternionic structure* on M is a rank-3 subbundle $P \subset \text{End}(TM)$ which is locally spanned by an almost hyper-paracomplex structure $\tilde{H} = (J_\alpha)$; such a locally defined triple \tilde{H} will be called admissible basis of P . A linear connection D on TM is called *paraquaternionic connection* if D preserves P , i.e. there exist locally defined 1-forms $\omega_\alpha, \alpha = 1, 2, 3$ such that

$$(6.45) \quad DJ_1 = -\omega_3 \otimes J_2 + \omega_2 \otimes J_3, \quad DJ_2 = \omega_3 \otimes J_1 + \omega_1 \otimes J_3, \quad DJ_3 = \omega_2 \otimes J_1 + \omega_1 \otimes J_2.$$

Consequently, the curvature R^D of D satisfies the relations

$$(6.46) \quad \begin{aligned} [R^D, J_1] &= -A_3 \otimes J_2 + A_2 \otimes J_3, \\ [R^D, J_2] &= A_3 \otimes J_1 + A_1 \otimes J_3, \\ [R^D, J_3] &= A_2 \otimes J_1 + A_1 \otimes J_2, \end{aligned}$$

$$A_1 = d\omega_1 + \omega_2 \wedge \omega_3, \quad A_2 = d\omega_2 + \omega_3 \wedge \omega_1, \quad A_3 = d\omega_3 - \omega_1 \wedge \omega_2.$$

An almost paraquaternionic structure is said to be a *paraquaternionic* if there is a torsion-free paraquaternionic connection.

A *P-Hermitian metric* is a pseudo Riemannian metric which is compatible with the (almost) hyper-paracomplex structure $\tilde{H} = (J_\alpha), \alpha = 1, 2, 3$ in the sense that the metric g is skew-symmetric with respect to each $J_\alpha, \alpha = 1, 2, 3$, i.e.

$$(6.47) \quad g(J_1 \cdot, J_1 \cdot) = g(J_2 \cdot, J_2 \cdot) = -g(J_3 \cdot, J_3 \cdot) = -g(\cdot, \cdot).$$

The metric g is necessarily of neutral signature $(2n, 2n)$. Such a structure is called (*almost*) *hyper-paraHermitian structure*.

An almost paraquaternionic (resp. paraquaternionic) manifold with P-Hermitian metric is called an *almost paraquaternionic Hermitian* (resp. *paraquaternionic Hermitian*) manifold. If the Levi-Civita connection of a paraquaternionic Hermitian manifold is paraquaternionic connection, then the manifold is said to be *paraquaternionic Kähler* manifold. This condition is equivalent to the statement that the holonomy group of g is contained in $Sp(n, \mathbb{R})Sp(1, \mathbb{R})$ for $n \geq 2$ [31, 64]. A typical example is the paraquaternionic projective space endowed with the standard paraquaternionic Kähler structure [20]. Any paraquaternionic Kähler manifold of dimension $4n \geq 8$ is known to be Einstein with scalar curvature s [31, 64]. If on a paraquaternionic Kähler manifold there exists an admissible basis (\tilde{H}) such that each $J_\alpha, \alpha = 1, 2, 3$ is parallel with respect to the Levi-Civita connection, then the manifold is said to be *hyper-paraKähler*. Such manifolds are also called *hypersymplectic* [39], *neutral hyper-Kähler* [46, 29]. The equivalent characterization is that the holonomy group of g is contained in $Sp(n, \mathbb{R})$ if $n \geq 2$ [64].

When $n \geq 2$, the paraquaternionic condition, i.e. the existence of torsion-free paraquaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, \tilde{H})Sp(1, \mathbb{R}) \cong GL(2n, \mathbb{R})Sp(1, \mathbb{R})$ -structure [2, 4]. Such a structure is a type of a para-conformal structure [9] as well as a type of generalized hypercomplex structure [17].

6.2. Hyper-paracomplex structures on 4-manifold. For $n = 1$ an almost paraquaternionic structure is the same as an oriented neutral conformal structure and turns out to be always paraquaternionic [26, 31, 64, 22]. The existence of a (local) hyper-paracomplex structure is a strong condition because of the next

Theorem 6.2. *If on a 4-manifold there exists a (local) hyper-paracomplex structure then the corresponding neutral conformal structure is anti-self-dual.*

Proof. Let $(g, (J_\alpha), \alpha = 1, 2, 3)$ be an almost hyper-paraHermitian structure with fundamental 2-form F_α associated to each J_α . Denote by $\theta_1, \theta_2, \theta_3$ the corresponding Lee forms (defined by $\theta_\alpha = -\delta F_\alpha \circ J_\alpha^3$).

Lemma 6.3. *The structure $(g, (J_\alpha), \alpha = 1, 2, 3)$ is a hyper-paracomplex structure, if and only if the three Lee forms coincide, $\theta_1 = \theta_2 = \theta_3$.*

Proof. The Levi-Civita connection satisfies (6.45). Consequently the Nijenhuis tensors obey

$$(6.48) \quad N_\alpha = -B_\alpha \otimes J_\beta + J_\beta \otimes B_\alpha - J_\alpha B_\alpha \otimes J_\gamma + J_\gamma \otimes J_\alpha B_\alpha, \quad B_\alpha = \omega_\beta - J_\alpha^3 \omega_\gamma.$$

Simple calculations using (6.45) give

$$\theta_1 = -J_2 \omega_2 + J_3 \omega_3, \quad \theta_2 = J_1 \omega_1 + J_3 \omega_3, \quad \theta_3 = -J_2 \omega_2 + J_1 \omega_1.$$

The last three identities and (6.48) yield

$$J_1(\theta_2 - \theta_1) = B_3, \quad J_2(\theta_2 - \theta_3) = B_1, \quad J_3(\theta_3 - \theta_1) = B_2.$$

Another glance at (6.48) completes the proof of the lemma. \square

Suppose that each $J_\alpha, \alpha = 1, 2, 3$ is integrable. Denote the common Lee form by θ and take the 3-form T to be the Hodge-dual to θ with respect to g . We have the identities $T = *\theta = -\theta \circ J_1 \wedge F_1 = -\theta \circ J_2 \wedge F_2 = +\theta \circ J_3 \wedge F_3$. Then the Bismut connections of the three structures coincide, i.e. the linear connection $\nabla^b := \nabla^g + \frac{1}{2}T$ preserves the metric and each $J_\alpha, \alpha = 1, 2, 3$. Therefore, each fundamental two form is parallel with respect to this connection, $\nabla^b F_\alpha = 0, \alpha = 1, 2, 3$. Consequently, the 2-form $\Phi_1 = F_2 + F_3$ is ∇^b -parallel,

$\nabla^b \Phi_1 = 0$. The 2-form Φ_1 is a $(2,0)+(0,2)$ -form with respect to J_1 . Hence, the Ricci form of the Bismut connection vanishes and $W^+ = 0$ due to Proposition 4.3 \square

Note that the integrability condition, Lemma 6.3, in the case of hyper-complex structure is due to F.Battaglia and S.Salamon (see [34]).

The universal cover $\widetilde{SL(2, \mathbb{R})}$ of the Lie group $SL(2, \mathbb{R})$ admits a discrete subgroup Γ such that the quotient space $(\widetilde{SL(2, \mathbb{R})}/\Gamma)$ is a compact 3-manifold [54, 59, 63]. Such a space has to be Seifert fibre space [63] and all the quotients are classified in [59]. The compact 4-manifold $M = S^1 \times (\widetilde{SL(2, \mathbb{R})}/\Gamma)$ admits a complex structure and is known as Kodaira-Thurston surface modeled on $S^1 \times \widetilde{SL(2, \mathbb{R})}$ [65].

6.3. Proof of Theorem 1.3. Let $S_2^2(1) = \{\mathbb{R}^4 \ni (a, b, c, d) : a^2 + b^2 - c^2 - d^2 = 1\}$ be the unit pseudo-sphere with respect to the standart neutral metric in \mathbb{R}^4 . We consider the so-called *hyperbolic Hopf manifold* $\mathbb{R} \times S_2^2(1)$ isomorphic to the Lie group $\mathbb{R} \times SL(2, \mathbb{R})$. The Lie algebra $\mathbb{R} \times sl(2, \mathbb{R}) \cong gl(2, \mathbb{R})$ has a basis $\{W, X, Y, Z\}$ with Z central and non-zero brackets given by

$$[X, Y] = W, \quad [Y, W] = -X, \quad [W, X] = Y.$$

An almost paracomplex structure on $\mathbb{R} \times SL(2, \mathbb{R})$ is constructed in [13]. The Lie algebra $\mathbb{R} \times sl(2, \mathbb{R})$ supports a hyper-paracomplex structure given by [4, 22]

$$J_3 Z = X, \quad J_3 Y = W, \quad J_2 Z = Y, \quad J_2 X = -W.$$

We pick a compatible neutral metric g , in the corresponding conformal class, defined such that the basis $\{W, X, Y, Z\}$ is an orthonormal basis, X, Z have norm 1 while Y, W have norm -1 , $g(X, X) = g(Z, Z) = -g(W, W) = -g(Y, Y) = 1$.

Lemma 6.4. *The invariant hyper-paraHermitian structure on $\mathbb{R} \times SL(2, \mathbb{R})$, described above, is non-flat conformally equivalent to a flat hyper-paraKähler structure.*

More precisely, the Lee form $\theta = -Z$ is ∇^g -parallel and the complex product connection coincides with the Levi-Civita connection of the flat hyper-paraKähler metric $g^{ob} = e^{-t}g$, where t is the local coordinate on \mathbb{R} .

Proof. The Koszul formula gives the following non-zero terms:

$$\begin{aligned} 2\nabla_X^g Y &= W, & 2\nabla_Y^g W &= -X, & 2\nabla_W^g X &= Y, \\ 2\nabla_X^g W &= -Y, & 2\nabla_Y^g X &= -W, & 2\nabla_W^g Y &= X. \end{aligned}$$

It is easy to check that g is not flat and $\theta = -Z = -dt$ satisfies $\nabla^g \theta = 0$. The Levi-Civita connection of the conformal metric $g' = e^{-t}g$ is determined by

$$2\nabla_A^{g'} B := 2\nabla_A^g B - \theta(A)B - \theta(B)A + g(A, B)\theta.$$

It is straightforward to verify that $\nabla^{g'}$ preserves J_1, J_2, J_3 . Hence, it is the complex product connection and the metric g' is hyper-paraKähler. It is not difficult to calculate that the connection $\nabla^{g'}$ is flat which proves the lemma. \square

An (left) invariant Weyl-flat hyper-paraHermitian structure on $\mathbb{R} \times SL(2, \mathbb{R})$ is just the neutral product of the standard Lorentz metric of constant sectional curvature on the unit pseudo-sphere $S_2^2(1)$, induced by the neutral metric on \mathbb{R}^4 , and the flat metric on \mathbb{R} . In coordinates (x, y, z, t) , it has the form

$$ds^2 = (\cosh y)^2 (\cosh z)^2 dx dx + dt dt - (\cosh z)^2 dy dy - dz dz.$$

The left-invariant Weyl-flat hyper-paraHermitian structure on $\mathbb{R} \times \widetilde{SL(2, \mathbb{R})}$ described in Lemma 6.4 descends to $M = S^1 \times (\widetilde{SL(2, \mathbb{R})}/\Gamma)$. The descended structure is not globally conformal to a hyper-paraKähler structure since the closed Lee form θ is actually a 1 form on

the circle S^1 and therefore can not be exact. Hence, the proof of Theorem 1.3 is completed. \square

The 4-dimensional Lie algebras admitting a hyper-paracomplex structure were classified recently in [22]. It is shown in [22] that exactly 10 types of Lie algebras admit a hyper-paracomplex structure. Theorem 6.2 tells us that the corresponding neutral metrics are anti-self-dual. We show below that some of them are not conformally flat.

Note that all 4-dimensional Lie groups admitting anti-self-dual non Weyl-flat Riemannian metric are classified in [25].

Example 6.5. We recall the construction of hyper-paracomplex structures on some 4-dimensional Lie algebras keeping the notations in [22].

- i) Consider the solvable Lie algebra PHC5 with a basis $\{X, Y, Z, W\}$, non-zero bracket $[X, Y] = X$ and hyper-paracomplex structure given by

$$J_3Z = W, \quad J_3X = Y, \quad J_2Z = W, \quad J_2X = Y - Z, \quad J_2Y = X + W.$$

Consider the oriented basis $A = X, \quad B = Y, \quad C = Y - Z, \quad D = -X - W$ and pick a compatible neutral metric g with non-zero values on the basis $\{A, B, C, D\}$ given by $g(A, A) = g(B, B) = -g(C, C) = -g(D, D) = 1$. The metric g on the corresponding simply connected solvable Lie group is conformally hyper-paraKähler (hypersymplectic) since the Lee form $\theta = B - C$ is closed and therefore exact. It is anti-self-dual metric with non-zero Weyl tensor because its curvature $R^g(A, B, C, D) = 1$. In local coordinates $\{x, y, z, t\}$, the metric is given by

$$ds^2 = e^{2y} dx dx + dy dy - e^{-y} (dx dt + dt dx) + (dy dz + dz dy).$$

- ii) Consider the solvable Lie algebras PHC6, PHC9, PHC10 defined by non-zero brackets:

$$\text{PHC6} \quad [X, Y] = Z, [X, W] = X + aY + bZ, [W, Y] = Y$$

$$\text{PHC9} \quad [Z, W] = Z, [X, W] = cX + aY + bZ, [Y, W] = Y, c \neq 0$$

$$\text{PHC10} \quad [Y, X] = Z, [W, Z] = cZ, [W, X] = \frac{1}{2}X + aY + bZ, [W, Y] = (c - \frac{1}{2})Y, c \neq 0$$

These algebras admit a hyper-paracomplex structure defined by

$$J_3Z = Y, \quad J_3X = W, \quad J_2Z = Y, \quad J_2X = W - Z, \quad J_2W = X + Y.$$

Consider the oriented frame $A = X, B = W, C = W - Z, D = -X - Y$. A compatible metric g is defined such that the frame $\{A, B, C, D\}$ is orthonormal with $g(A, A) = g(B, B) = -g(C, C) = -g(D, D) = 1$. The Lee forms of these hyper-paraHermitian structures are closed and the curvature satisfies

$$\text{PHC6} \quad R^g(A, B, C, D) = (1 - a);$$

$$\text{PHC9} \quad R^g(A, B, C, D) = \frac{1}{2}(2c^2 - 3c - 2ac + 2a + 1);$$

$$\text{PHC10} \quad R^g(A, B, C, D) = \frac{1}{2}(c^2 + 2ac - c);$$

Clearly there are constants (a, b, c) such that the corresponding Lie algebras admit anti-self-dual neutral metric with non-zero Weyl tensor. For example, let us take $c = -2, \quad a = b = 0$ in the Lie algebra PHC9 described in Example 6.5, ii). The Lee form $\theta = B - C$ is not ∇^g -parallel but closed and the Weyl curvature does not vanish because $R(A, B, C, D) = 15/2$. In coordinates x, y, z, t the left invariant vector fields A, B, C, D can be expressed as follows

$$A = e^{-2t} \frac{\partial}{\partial x}, \quad B = \frac{\partial}{\partial t}, \quad C = \frac{\partial}{\partial t} - e^t \frac{\partial}{\partial z}, \quad D = -e^{-2t} \frac{\partial}{\partial x} - e^t \frac{\partial}{\partial y}.$$

The invariant neutral anti-self-dual metric with non-zero Weyl tensor has the form

$$ds^2 = e^{4t} dx dx + dt dt - e^t (dx dy + dy dx) + e^{-t} (dt dz + dz dt).$$

It turns out that the conformal structure $[g]$ induced by the invariant hyper-paracomplex structure on the corresponding simply connected 4-dimensional Lie group is actually generated by a hyper-paraKähler (hypersymplectic) structure, since the Lee form θ is closed (and therefore exact) in all 10 possible cases described in [22]. On some of them the Lee form is zero and the structure is hyper-paraKähler (hypersymplectic).

Example 6.6. The solvable Lie group corresponding to the Lie algebra defined in [4] and obtained from *PHC9* for $c = -1$, $a = b = 0$, possesses an invariant hyper-paraKähler (hypersymplectic) structure with non-zero Weyl tensor since the Lee form vanishes and the curvature has non-zero value on an orthonormal basis.

Summarizing, we get

Proposition 6.7. *Any one of the nine simply connected solvable Lie groups corresponding to a solvable 4-dimensional Lie algebra admitting hyper-paracomplex structure supports a hyper-paraKähler (hypersymplectic) structure.*

Remark 6.8. The hyper-paraKähler (hypersymplectic) structures on the nine solvable Lie groups mentioned in Proposition 6.7, are not left-invariant in general. There are left-invariant hypersymplectic structure on exactly four cases according to the recent classification of the hypersymplectic 4-dimensional Lie algebras [3].

Due to the Malcev theorem [52], the 4-dimensional nilpotent Lie group H has a discrete subgroup Γ such that the quotient $M = H/\Gamma$ is a compact nil-manifold, the Kodaira surface. It is known that these surfaces admit a hyper-paraKähler (hypersymplectic) structure [46], see also [29].

Consider the solvable Lie algebra sol_1^4 defined by non-zero brackets:

$$[X, Y] = Z, [X, W] = X, [W, Y] = Y.$$

This Lie algebra can be obtained by taking $a = b = 0$ in the Lie algebra *PHC6* described in Example 6.5, ii).

The corresponding solvable Lie group is known to be Sol_1^4 . The geometric structures modeled on this group appear as one of the possible geometric structures on 4-manifold [65]. The compact quotients of Sol_1^4 by a discrete group Γ constitute the Inoue surfaces modeled on Sol_1^4 [65].

6.4. Proof of Theorem 1.4. A hyper-paracomplex structure on the Lie algebra sol_1^4 is given by

$$J_3 Z = Y, \quad J_3 X = W, \quad J_2 Z = Y, \quad J_2 X = W - Z, \quad J_2 W = X + Y.$$

Consider the oriented frame $A = X, B = W, C = W - Z, D = -X - Y$. A compatible metric g is defined such that the frame $\{A, B, C, D\}$ is orthonormal with $g(A, A) = g(B, B) = -g(C, C) = -g(D, D) = 1$. The Lee form $\theta = B - C$ is not ∇^g -parallel but closed and the Weyl curvature does not vanish because $R(A, B, C, D) = 1$.

In coordinates x, y, z, t , the left invariant vector fields A, B, C, D on Sol_1^4 can be expressed as follows

$$A = e^{-t} \frac{\partial}{\partial x}, \quad B = \frac{\partial}{\partial t}, \quad C = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}, \quad D = -e^{-t} \frac{\partial}{\partial x} - e^t \frac{\partial}{\partial y} - e^t x \frac{\partial}{\partial z}.$$

The left invariant neutral anti-self-dual metric with non-zero Weyl tensor on Sol_1^4 has the form

$$ds^2 = e^{2t} dx dx + dt dt - (dx dy + dy dx) + (dt dz + dz dt) - x(dt dy + dy dt).$$

The left invariant hyper-paracomplex structures on Sol_1^4 , described above, descends to the Inoue surfaces modeled on Sol_1^4 . Theorem 6.2 completes the proof of Theorem 1.4. \square

All local hyper-paracomplex structures on 4-manifold, we have presented, are locally

conformal to hypersymplectic structures. We shall construct a local hyper-paracomplex structure which is not conformally equivalent to a hyper-paraKähler (hypersymplectic), i.e. its Lee form $d\theta \neq 0$. We adapt the Ashtekar at all [7] formulation of the self-duality Einstein equations to the case of neutral metric and modify the Joyce's construction [45] of hyper-complex structure from holomorphic functions.

Example 6.9. Let V_1, V_2, V_3, V_4 be a vector fields on an oriented 4-manifold M forming an oriented basis for TM at each point. Then V_1, \dots, V_4 define a neutral conformal structure $[g]$ on M . Define an almost hyper-paracomplex structure (J_2, J_3) by the equations

$$J_3 V_1 = -V_2, \quad J_3 V_3 = V_4, \quad J_2 V_1 = -V_4, \quad J_2 V_2 = V_3.$$

Suppose that V_1, \dots, V_4 satisfy the three vector field equations

$$(6.49) \quad [V_1, V_2] + [V_3, V_4] = 0, \quad [V_1, V_3] + [V_2, V_4] = 0, \quad [V_1, V_4] - [V_2, V_3] = 0.$$

It is easy to check that these equations imply the integrability of (J_2, J_3) , i.e. (J_2, J_3) is a hyper-paracomplex structure which is compatible with the neutral conformal structure $[g]$. Hence, $[g]$ is anti-self-dual, due to Theorem 6.2.

The neutral Ashtekar at all equation (6.49) may be written in a complex form

$$(6.50) \quad [V_1 + iV_2, V_1 - iV_2] + [V_3 + iV_4, V_3 - iV_4] = 0, \quad [V_1 + iV_2, V_3 - iV_4] = 0.$$

Let M be a complex surface, let (z^1, z^2) be local holomorphic coordinates, and define V_1, \dots, V_4 by

$$V_1 + iV_2 = f_1 \frac{\partial}{\partial z^1} + f_2 \frac{\partial}{\partial z^2}, \quad V_3 + iV_4 = f_3 \frac{\partial}{\partial z^1} + f_4 \frac{\partial}{\partial z^2},$$

where f_j is a complex function on M . Substituting into (6.50) we find the equations are satisfied identically if f_j is a holomorphic function with respect to the complex structure on M . So we can construct a hyper-paracomplex structure, with the opposite orientation, out of four holomorphic functions f_1, \dots, f_4 .

Taking $f_1 = f, f_2 = f_3 = 0, f_4 = 1$ we obtain a local hyper-paracomplex structure. Consider a particular neutral metric $g \in [g]$ such that

$$g(V_1, V_1) = g(V_2, V_2) = -g(V_3, V_3) = -g(V_4, V_4) = 1, g(V_j, V_k) = 0, j \neq k.$$

The corresponding common Lee form is given by

$$\theta = \frac{1}{f} \frac{\partial f}{\partial z^2} dz^2 + \frac{1}{\bar{f}} \frac{\partial \bar{f}}{\partial \bar{z}^2} d\bar{z}^2.$$

Then $d\theta \neq 0$ provided $\frac{\partial f}{\partial z^1} \neq 0$.

6.5. Twistor and reflector spaces on paraquaternionic Kähler manifold. Consider the space \tilde{H}_1 of imaginary para-quaternions. It is isomorphic to the Lorentz space \mathbb{R}_1^2 with a Lorentz metric of signature $(+, +, -)$ defined by $\langle q, q' \rangle = -\text{Re}(q\bar{q}')$, where $\bar{q} = -q$ is the conjugate imaginary paraquaternion. In \mathbb{R}_1^2 there are two kinds of 'unit spheres', namely the pseudo-sphere $S_1^2(1)$ of radius 1 (the 1-sheeted hyperboloid) which consists of all imaginary para-quaternions of norm 1 and the pseudo-sphere $S_1^2(-1)$ of radius (-1) (the 2-sheeted hyperboloid) which contains all imaginary para-quaternions of norm (-1). The 1-sheeted hyperboloid $S_1^2(1)$ carries a natural paraHermitian structure while the 2-sheeted hyperboloid $S_1^2(-1)$ carries a natural Hermitian structure of signature (1,1), both induced by the restriction of the Lorentz metric and the cross-product on $\tilde{H}_1 \cong \mathbb{R}_1^2$ defined by

$$X \times Y = \sum_{i \neq k} x^i y^k J_i J_k$$

for vectors $X = x^i J_i$, $Y = y^k J_k$. Namely, for a tangent vector $X = x^i J_i$ to the 1-sheeted hyperboloid $S_1^2(1)$ at a point $q_+ = q_+^k J_k$ (resp. tangent vector $Y = y_-^k J_k$ to the 2-sheeted hyperboloid $S_1^2(-1)$ at a point $q_- = q_-^k J_k$) we define $PX := q_+ \times X$ (resp. $JY = q_- \times Y$).

It is easy to check that PX is again tangent vector to $S_1^2(1)$, $P^2X = X$, $\langle PX, PX \rangle = -\langle X, X \rangle$ (resp. JY is tangent vector to $S_1^2(-1)$, $J^2Y = -Y$, $\langle JY, JY \rangle = \langle Y, Y \rangle$).

We start with a paraquaternionic Kähler manifold $(M, g, \tilde{H} = (J_\alpha))$. The vector bundle P carries a natural Lorentz structure of signature $(+, +, -)$ such that (J_1, J_2, J_3) forms an orthonormal local basis of P . There are two kinds of "unit sphere" bundles according to the existence of the 1-sheeted hyperboloid $S_1^2(1)$ and the 2-sheeted hyperboloid $S_1^2(-1)$. The twistor space $Z^+(M)$ (resp. $Z^-(M)$) is the unit pseudo-sphere bundle with fibre $S_1^2(1)$ (resp. $S_1^2(-1)$). In other words, the fibre of $Z^+(M)$ consists of all almost paracomplex structures (resp. all almost complex structures) compatible with the given paraquaternionic Kähler structure. The bundle $Z^+(M)$ over a 4-dimensional manifold with a neutral metric was constructed in [44] and called there *the reflector space*. Further, we keep their notation.

Denote by π^\pm the projection of $Z^\pm(M)$ onto M , respectively. Keeping in mind the formal similarity with the quaternionic geometry where there are two natural almost complex structures [8, 27], we observe the existence of two naturally arising almost paracomplex structures on $Z^+(M)$ (resp. two almost complex structures on $Z^-(M)$) [19] defined as follows:

The Levi-Civita connection on P preserves the Lorentz metric and induces a linear connection on Z^\pm i.e a splitting of the tangent bundle $TZ^\pm = \mathbb{H}^\pm \otimes \mathbb{V}^\pm$, respectively, where \mathbb{V}^\pm is the vertical distribution tangent to the fibre $S_1^2(1)$, (resp. $S_1^2(-1)$) and \mathbb{H}^\pm a supplementary horizontal distribution induced by the Levi-Civita connection. By definition, the horizontal transport associated to \mathbb{H}^\pm preserves the canonical Lorentz metric of the fibres $S_1^2(1)$ (resp. $S_1^2(-1)$); and also their orientation; as a corollary, it preserves the canonical paracomplex structure on $S_1^2(1)$ (resp. the canonical complex structure on $S_1^2(-1)$) described above. Since the vertical distribution \mathbb{V}^\pm is tangent to the fibres, this paracomplex structure (resp. complex structure) induces an endomorphism \tilde{P} with $\tilde{P}^2 = id$ (resp. \tilde{J} with $\tilde{J}^2 = -id$) on \mathbb{V}_z^\pm for each $z \in Z^+(M)$ (resp. $Z^-(M)$). On the other hand, each point z on $Z^+(M)$ (resp. $Z^-(M)$) is by definition a paracomplex structure on $T_{\pi(z)}Z^+(M)$ (resp. a complex structure on $T_{\pi(z)}Z^-(M)$) which may be lifted into an endomorphism \bar{P} on \mathbb{H}_z^+ with $\bar{P}^2 = id$ (resp. \bar{J} on \mathbb{H}_z^- with $\bar{J}^2 = -id$). We define almost paracomplex structures $\mathbb{P}_1, \mathbb{P}_2$ on $Z^+(M)$ and almost complex structures $\mathbb{J}_1, \mathbb{J}_2$ on $Z^-(M)$ by

$$\begin{aligned} \mathbb{P}_1(\mathbb{V}^+) &= \mathbb{V}^+, & \mathbb{P}_1|_{\mathbb{V}^+} &= \tilde{P}, & \mathbb{P}_1(\mathbb{H}^+) &= \mathbb{H}^+, & \mathbb{P}_1|_{\mathbb{H}^+} &= \bar{P}, \\ \mathbb{P}_2(\mathbb{V}^+) &= \mathbb{V}^+, & \mathbb{P}_2|_{\mathbb{V}^+} &= -\tilde{P}, & \mathbb{P}_2(\mathbb{H}^+) &= \mathbb{H}^+, & \mathbb{P}_2|_{\mathbb{H}^+} &= \bar{P}; \\ \mathbb{J}_1(\mathbb{V}^-) &= \mathbb{V}^-, & \mathbb{J}_1|_{\mathbb{V}^-} &= \tilde{J}, & \mathbb{J}_1(\mathbb{H}^-) &= \mathbb{H}^-, & \mathbb{J}_1|_{\mathbb{H}^-} &= \bar{J}, \\ \mathbb{J}_2(\mathbb{V}^-) &= \mathbb{V}^-, & \mathbb{J}_2|_{\mathbb{V}^-} &= -\tilde{J}, & \mathbb{J}_2(\mathbb{H}^-) &= \mathbb{H}^-, & \mathbb{J}_2|_{\mathbb{H}^-} &= \bar{J}. \end{aligned}$$

Define pseudo Riemannian metrics on $Z^+(M)$ (resp. $Z^-(M)$) by $h_t^+ = \pi^*g + t\langle \cdot, \cdot \rangle_v^+$, $t \neq 0, \langle \cdot, \cdot \rangle_v^+$ being the restriction of the Lorentz metric to the fibres $S_1^2(1)$ (resp. $h_t^- = \pi^*g + t\langle \cdot, \cdot \rangle_v^-$, $t \neq 0, \langle \cdot, \cdot \rangle_v^-$ being the restriction of the Lorentz metric to the fibres $S_1^2(-1)$). It is easy to check that h^+ (resp. h^-) is compatible with both $\mathbb{P}_1, \mathbb{P}_2$ (resp. $\mathbb{J}_1, \mathbb{J}_2$) such that $(Z^+(M), h_t^+, \mathbb{P}_{1,2})$ become almost paraHermitian manifolds (resp. $(Z^-(M), h_t^-, \mathbb{J}_{1,2})$ become almost Hermitian manifolds).

The almost paracomplex structures $\mathbb{P}_1, \mathbb{P}_2$ and the neutral metrics h_t^+ on the reflector space of a 4-dimensional manifold with a neutral metric g are investigated in [44]. The authors show that the almost paracomplex structure \mathbb{P}_2 is never integrable while the almost paracomplex structure \mathbb{P}_1 is integrable if and only if the neutral metric g is self dual. They also prove that the neutral metric h_t^+ on the reflector space is Einstein if and only if g is self-dual Einstein and either $ts = 12$ or $ts = 6$.

Almost Hermitian geometry of $(Z^-(M), h_t^-, \mathbb{J}_{1,2})$ is investigated in [19]. The calculations there are completely applicable to the almost paraHermitian geometry of $(Z^+(M), h_t^+, \mathbb{P}_{1,2})$. In terms of the almost paraHermitian geometry of $(Z^+(M), h_t^+, \mathbb{P}_{1,2})$ Theorem 1 and Theorem 2 in [19] read as follows

Theorem 6.10. *On the reflector space $(Z^+(M))$ of a paraquaternionic Kähler manifold of dimension $4n \geq 8$ we have:*

- i) *The almost paracomplex structure \mathbb{P}_1 is integrable and the Lee form of the paraHermitian structure (\mathbb{P}_1, h_t^+) is zero. The structure (\mathbb{P}_1, h_t^+) is paraKähler if and only if $ts = 4n(n+2)$;*
- ii) *The almost paracomplex structure \mathbb{P}_2 is never integrable and the Lee form of the almost paraHermitian structure (\mathbb{P}_2, h_t^+) is zero. The structure (\mathbb{P}_2, h_t^+) is nearly paraKähler if and only if $ts = 2n(n+2)$ and almost paraKähler if and only if $ts = -4n(n+2)$.*

Theorem 6.11. *On the reflector space $(Z^+(M))$ of an oriented 4-dimensional manifold M with a neutral metric g we have the following:*

- i) *The almost paraHermitian structure (\mathbb{P}_1, h_t^+) has zero Lee form if and only if the metric g is self-dual. It is paraKähler if and only if the metric g is Einstein self-dual and $ts = 12$;*
- ii) *The Lee form of the almost paraHermitian structure (\mathbb{P}_2, h_t^+) is zero. The structure (\mathbb{P}_2, h_t^+) is nearly paraKähler if and only if the metric g is self-dual Einstein and $ts = 6$ and almost paraKähler if and only if $ts = -12$.*

Remark 6.12. On a paraquaternionic manifold of dimension $4n \geq 8$ we may construct the almost paracomplex structure \mathbb{P}_1 on the reflector space Z^+ and the almost complex structure \mathbb{J}_1 on the twistor space Z^- using the horizontal distribution generated by a torsion-free connection instead of the horizontal distribution of the Levi-Civita connection. In that case, we find an analogue of the result of S. Salamon [60, 61, 62], (proved also independently by L. Berard-Bergery, unpublished, see [16]). Namely, we have

Theorem 6.13. *On a paraquaternionic manifold of dimension $4n \geq 8$ the almost paracomplex structure \mathbb{P}_1 on Z^+ and the almost complex structure \mathbb{J}_1 on Z^- are always integrable*

We sketch a proof which is completely similar to the proof in the case of quaternionic manifold presented in [16]. Denote by R the curvature of a torsion-free connection ∇ . Let $S = xJ_1 + yJ_2 + zJ_3$ be either an almost paracomplex structure or an almost complex structure compatible with the given paraquaternionic structure i.e. the triple (x, y, z) satisfies either $x^2 + y^2 - z^2 = 1$ or $x^2 + y^2 - z^2 = -1$. Denote by

$$S(R)(X, Y) = [R(SX, SY), S] - S[R(SX, Y), S] - S[R(X, SY), S] + S^2[R(X, Y), S].$$

In view of the analogy with the proof in the quaternionic case presented in [16], 14.72-14.74 the result will follow if $S(R) = 0$. The last identity can be checked in the exactly same way as it is done in [16], Lemma 14.74 using (6.1) instead of formulas 14.39 in [16].

6.6. Examples of nearly paraKähler and almost paraKähler manifolds. Theorem 6.10 helps to find examples of nearly paraKähler and almost paraKähler manifolds. We note that the sign of the scalar curvature (if not zero) is not a restriction since the metric (-g) have scalar curvature with opposite sign. Hence, taking the reflector space of any neutral (anti) self-dual Einstein manifold with non-zero scalar curvature in dimension four and any quaternionic paraKähler manifold in dimension $4n \geq 8$ we can find a real number t to get nearly paraKähler and almost paraKähler structure on it.

- (1) The pseudo-sphere S_3^3 is endowed with an almost paracomplex structure [51] and it is shown in [13] that there exists a nearly paraKähler structure on S_3^3 induced from the so-called *second kind Cayley numbers* (see [51]) in \mathbb{R}_3^4 . The structure is Einstein with non-zero scalar curvature, in fact the metric is the standard neutral metric on S_3^3 inherited from \mathbb{R}_3^4 .
- (2) Start with one of the following 4-dimensional neutral self-dual Einstein spaces $(S_2^2 = SO^+(2, 3)/GL^+(2, \mathbb{R}), can)$, $(\mathbb{CP}^{1,1} = (SU(2, 1)/(SO(1, 1).U(1)), can)$, or $(SL(3, \mathbb{R})/GL^+(2, \mathbb{R}), c.Kill|_{sl(3, \mathbb{R})})$, where c is a suitable constant and $Kill|_{sl(3, \mathbb{R})}$ is the restriction of the Killing form of $sl(3, \mathbb{R})$ to the homogeneous space $SL(3, \mathbb{R})/GL^+(2, \mathbb{R})$. The corresponding reflector spaces are $SO^+(2, 3)/GL^+(2, \mathbb{R})$, $SU(2, 1)/(SO(1, 1).U(1))$, $SL(3, \mathbb{R})/(\mathbb{R}^+ \times \mathbb{R}^+ \cup \mathbb{R}^- \times \mathbb{R}^-)$, respectively [48]. These homogeneous spaces admit a homogeneous nearly paraKähler structure of non-zero scalar curvature as well as an homogeneous almost paraKähler structure according to Theorem 6.11
- (3) Non-homogeneous example arises from the non-(locally) homogeneous neutral self-dual Einstein space of non-zero scalar curvature described in [21]. Its reflector space admit a nearly paraKähler structure of non-zero scalar curvature, as well as an almost paraKähler structure, due to Theorem 6.11

To the best of our knowledge there are no known examples of Ricci flat 6-dimensional Nearly paraKähler manifolds.

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